International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 6, Issue 4, Jun – Jul 2017; 107-114

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

ON CONVERGENCE PROPERTIES OF SZASZ TYPE POSITIVE LINEAR OPERATOR

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ABSTRACT

In this paper, we have given Szasz type, positive linear operator and we have shown that, these operators preserve convergence properties for n when, f is τ -convex.

2010 Mathematics Subject Classification: 41A25; 41A36

KEYWORDS: Linear Operators, Szasz- Mirakyan Operators, Bernstein Polynomials

INTRODUCTION

In 1987, Lupus [1] introduced the first q-analogue of Bernstein operators, after that, Phillips [2] introduced another q-generalization of the classical Bernstein polynomials, later, many generalizations of positive linear operators, based on q-integers were introduced and studied by several authors. Some are in [3–5], Bleimann et al. [6], Proposed a sequence of positive linear operators L_n, defined by

$$L_n = \frac{1}{(1+x)^n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) \binom{n}{k} x^k \ \forall\ x \geq 0,\ n \in N \quad \textbf{(1)}$$

for $f \in C\,[0, \infty)$, where $C\,[0, \infty)$ denote the space of all continuous and real valued functions, defined in $[0, \infty)$. Also the authors proved that, $L_n\,(f; x) \rightarrow (x)$ as $n \rightarrow \infty$ point wise on $[0,\infty)$ for any $f \in C_B\,[0,\infty)$,where

$f \in C_B\,[0, \infty)$ denote the space of all bounded functions from $C\,[0, \infty)$.

Now, we recall some notations from q-analysis, the q-integer $[n]$ and the q-factorial $[n]!$ are defined by,

$$[n] := [n]_q = \begin{cases} \frac{1-q^n}{1-q} & q \neq 1 \\ n & q = 1 \end{cases} \quad \text{For } n \in N,\ [0]! = 0, \quad \textbf{(2)}$$

$$[n]! := \begin{cases} [1]_q [2]_q \ldots\ldots [n]_q, & n = 1,2,\ldots\ldots \\ 1 & n = 0 \end{cases} \quad n \in N,\ [0]! = 1 \text{ where } q > 0 \quad \textbf{(3)}$$

for integers $n \geq r \geq 0$ the q- binomial coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q\,!}{[r]_q\,![n-r]_q\,!} \quad \textbf{(4)}$$

Aral and Dogru [10] constructed the q-Bleimann, Butzer and Hahn operators as

$$L_{n,q}\,(f; x) = \frac{1}{\ell n\,(x)} \sum_{k=0}^{n} f\left(\frac{[k]}{[n-k+1]q^k}\right) q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k \quad \textbf{(5)}$$

Where $\ell n\,(x) = \prod_{s-o}^{n-1} (1 + q^s x\,)$

and f is defined on the semi axis $[0,\infty)$.

In [9], the authors introduced a new generalization of Bernstein polynomials, denoted by $B\tau^n$ and defined as,

$$B\tau^n(\text{f;x}) = \sum_{k=0}^{n} \binom{n}{k} (f o\tau^{-1}) f\left(\frac{k}{n}\right) (1-\tau(\text{x}))^{n-k}\ \tau(\text{x})^k$$

Where is the n^{th} Bernstein polynomial, $f \in [0, 1]$, $x \in [0, 1]$

$\boldsymbol{\tau}$ is a function defined on [0, 1] and having the properties:

$\boldsymbol{\tau}$ is∞-times continuously differentiable on [0, 1].

$\boldsymbol{\tau}(0) = 0$, $\boldsymbol{\tau}(1) = 1$ and $\boldsymbol{\tau}'(\mathbf{x}) > 0$ on [0, 1].

These conditions ensure that, τ is strictly increasing and the inverse τ^{-1} of τ exists on [0, 1].

We recall by [11] some usual notations and definitions, which are essential for our work.

For $\mathbf{x} = (x_1, x_2) \in R^2$, $\mathbf{k} = (k_1, k_2) \in N_0^2$ and $n \in N$

we will write $|\mathbf{x}| := x_1 + x_2$, $\mathbf{x}^{\mathbf{k}} := x^{k1}\, x^{k2}$, $|\mathbf{k}| := k_1 + k_2$, $\mathbf{k}! := k_1!k_2!$

Now we define, a new generalization of q-Bleimann, Butzer, and Hahn operators for $f \in [0, \infty)$ by [12]

$$L_{n,q}\ (\text{f}; \tau(y)) = \frac{1}{\ell\text{n (y)}} \sum_{k=0}^{n} (f o\tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right) q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \tau(y)^k$$

Where $\ell\text{n (y)} = \prod_{s=o}^{n-1}(1+q^s\tau(y))$

and τ is a function, that is continuously differentiable of infinite order on, [0, ∞)such that $\tau(0) = 0$, $\tau(1) = 1$, and $in\ f_{x\in[0,\infty)}\ \tau'(x) \geq 1$

We define new positive operator $f \in [0, \infty)$, from above work

$$S_{n,}\ (\text{f}; \tau(x)) = e^{-n\tau(x)} \sum_{k=0}^{n} \frac{(n\tau(x))^k}{k!} (f o\tau^{-1}) \left(\frac{k}{n}\right)$$

and τ is a function that is continuously differentiable of infinite order on [0, ∞)such that $\tau(0) = 0$, $\tau(1) = 1$, and $inf_{x\in[0,\infty)}\ \tau'(x) \geq 1$

Definition 1

Let f, be a real valued function continuous defined on, D ⊆ R^2 and let τ, be a function satisfying the conditions (τ_1) and (τ_2) .

We say that, f is a Lipchitz continuous function of order on D, if

$$|\text{f(x)} - \text{f (y)}| \leq |\sum_{i=1}^{2} |\tau(x_i) - \tau(y_i)|^{\mu}$$

Definition 2

A continuous real valued function f, is said to be convex in D ⊆ [0, ∞), if

$$\text{f}(\sum_{i=1}^{m} \propto_i x_i) \leq \sum_{i=1}^{m} \propto_i f(x_i)$$

for every $x_1 x_{2 \ldots\ldots\ldots,} x_m \in D$ and for every non negative number of $\alpha_1 \alpha_{2\ldots,} \alpha_m \in D$ such that, $\alpha_1 + \alpha_{2+\ \ldots\ldots\ldots+,} \alpha_m = 1$

MAIN RESULTS

Theorem: - Let τ −convex function defined on S. Then $S_{n,}$ (f; $\tau(x)$) is monotonically non –increasing in n.

Proof: Let x, y ∈ S and x ≤ y which means that $x_1 \leq y_1$ and $x_2 \leq y_2$. Using of the operators $s_{n,}^{\tau}$ and

From the definition $S_{n,}$ (f; $\tau(x)$), We have

$$S_{n,}(f;\tau(x)) = e^{-n\tau(x)} \sum_{k=0}^{n} \frac{(n\tau(x))^k}{k!}(fo\tau^{-1})\left(\frac{k}{n}\right)$$

$$S_{n,}(f;\tau(x)) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1} e^{-n\tau(x)}(\tau(x))^k \frac{(n))^k}{k!}(fo\tau^{-1})\left(\frac{k}{n}\right)$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}(|(\tau(x)|+1-|\tau(x)|)(\tau(x))^k \frac{(n))^k}{k!}(fo\tau^{-1})\left(\frac{k}{n}\right) e^{-n|\tau(x)|}$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1} \tau(x_1))^{k_1+1}\ \tau(x_2))^{k_2} \frac{(n))^k}{k!}(fo\tau^{-1})\left(\frac{k}{n}\right) e^{-n|\tau(x_1)-\tau(x_2)|}$$

$$+ \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}(\tau(x_1))^{k_1}\ (\tau(x_2))^{k_2+1} \frac{(n))^k}{k!}(fo\tau^{-1})\left(\frac{k}{n}\right) e^{-n|\tau(x_1)-\tau(x_2)|}$$

$$+ \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}(\tau(x))^k \frac{(n))^k}{k!}(fo\tau^{-1})\left(\frac{k}{n}\right) e^{-n|\tau(x)|}$$

Let $S_1 = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1} \tau(x_1))^{k_1+1}\ \tau(x_2))^{k_2} \frac{(n))^k}{k!}(fo\tau^{-1})\left(\frac{k}{n}\right) e^{-n|\tau(x_1)-\tau(x_2)|}$

$$S_2 = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}(\tau(x_1))^{k_1}\ (\tau(x_2))^{k_2+1} \frac{(n))^k}{k!}(fo\tau^{-1})\left(\frac{k}{n}\right) e^{-n|\tau(x_1)-\tau(x_2)|}$$

$$S_3 = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}(\tau(x))^k \frac{(n))^k}{k!}(fo\tau^{-1})\left(\frac{k}{n}\right) e^{-n|\tau(x)|}$$

Since,

$$S_1 = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1} \tau(x_1))^{k_1+1}\ \tau(x_2))^{k_2} \frac{(n)^k}{k!}(fo\tau^{-1})\left(\frac{k}{n}\right) e^{-n|\tau(x_1)-\tau(x_2)|}$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1} \tau(x_1))^{k_1+1}\ \tau(x_2))^{k_2}\left(\frac{(n)^k}{k!}\right) f(\tau^{-1}\left(\frac{k_1}{n_1}\right).\tau^{-1}\left(\frac{k_2}{n_2}\right)) e^{-n|\tau(x_1)-\tau(x_2)|}$$

$$+(\tau(x_1))^{n+1} f(\tau^{-1}(1).\tau^{-1}(0)) e^{-n|\tau(x_1)-\tau(x_2)|}$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1} \tau(x_1))^{k_1+1}\ \tau(x_2))^{k_2}\left(\frac{(n)^k}{k!}\right) f(\tau^{-1}\left(\frac{k_1}{n_1}\right).\tau^{-1}\left(\frac{k_2}{n_2}\right)) e^{-n|\tau(x_1)-\tau(x_2)|}$$

$$+ \sum_{k_1=0}^{n} \tau(x_1))^{k_1+1}\left(\frac{(n)^{k_1}}{k_1!}\right) f(\tau^{-1}\left(\left(\frac{k_1}{n_1}\right)\right).\tau^{-1}(0)) e^{-n|\tau(x_1)|}$$

$$+(\tau(x_1))^{n+1} f(\tau^{-1}(1).\tau^{-1}(0)) e^{-n|\tau(x_1)|}$$

$$= \sum_{k_1=0}^{n-2}\sum_{k_2=1}^{n-k_1-1} \tau(x_1))^{k_1+1}\ \tau(x_2))^{k_2} \frac{(n)^k}{k!} f(\tau^{-1}\left(\frac{k_1}{n_1}\right).\tau^{-1}\left(\frac{k_2}{n_2}\right) e^{-n|\tau(x_1)-\tau(x_2)|})$$

$$+ \sum_{k_1=0}^{n-1} \tau(x_1))^{k_1+1}\tau(x_2))^{n-k_1}\left(\frac{(n)^{k_1}}{k_1!}\right) f(\tau^{-1}\left(\frac{k_1}{n_1}\right).\tau^{-1}\left(\frac{n-k_1}{n_1}\right) e^{-n|\tau(x_1)-\tau(x_2)|}$$

$$+ \sum_{k_1=0}^{n-1} \tau(x_1))^{k_1+1}\left(\frac{(n)^k}{k!}\right) f(\tau^{-1}\left(\frac{k_1}{n_1}\right).\tau^{-1}((0))e^{-n|\tau(x_1)|}$$

$+(\tau(x_1))^{n+1}\ f\,(\tau^{-1}(1).\,\tau^{-1}(0))\ e^{-n|\tau(x_1)|}$

$=\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\tau(x))^k\ \frac{(n)^k}{k!}\ f(\tau^{-1}\left(\left(\frac{k_1-1}{n}\right)\right).\,\tau^{-1}\left(\left(\frac{k_2}{n}\right)\right)\ e^{-n|\tau(x)|}$

$+\sum_{k_1=1}^{n}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}\ (\frac{(n)^k}{k!})\ f(\tau^{-1}\left(\frac{k_1-1}{n}\right).\,\tau^{-1}\left(\left(\frac{n-k_1+1}{n}\right)\right)e^{-n|\tau(x_1)-\tau(x_2)|}$

$+\sum_{k_1=0}^{n}\tau(x_1))^{k_1}\ \frac{(n)^k}{k!}f(\tau^{-1}\left(\frac{k_1-1}{n}\right).\,\tau^{-1}((0)\ )e^{-n|\tau(x_1)|}$

$+(\tau(x_1))^{n+1}\ f\,(\tau^{-1}(1).\,\tau^{-1}(0))\ e^{-n|\tau(x_1)|}$

Since $\tau^{-1}(1)=1,\tau^{-1}(0)=0$

$S_1=\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\tau(x))^k\ \frac{(n)^k}{k!}\ f(\tau^{-1}\left(\left(\frac{k_1-1}{n}\right)\right).\,\tau^{-1}\left(\left(\frac{k_2}{n}\right)\right)\ e^{-n|\tau(x)|}$

$+\sum_{k_1=1}^{n}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}\ (\frac{(n)^k}{k!})\ f(\tau^{-1}\left(\frac{k_1-1}{n}\right).\,\tau^{-1}\left(\left(\frac{n-k_1+1}{n}\right)\right)e^{-n|\tau(x_1)-\tau(x_2)|}$

$+\sum_{k_1=0}^{n}\tau(x_1))^{k_1}\ \frac{(n)^k}{k!}f(\tau^{-1}\left(\frac{k_1-1}{n}\right).\,\tau^{-1}((0)\ )e^{-n|\tau(x_1)|}++(\tau(x_1))^{n+1}\ f\,(1,0)\ e^{-n|\tau(x_1)|}$

Similarly,

$S_2=\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\tau(x))^k\ \frac{(n)^k}{k!}\ f(\tau^{-1}\left(\left(\frac{k_1}{n}\right)\right).\,\tau^{-1}\left(\left(\frac{k_2-1}{n}\right)\right)\ e^{-n|\tau(x)|}$

$+\sum_{k_1=1}^{n}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}\ (\frac{(n)^k}{k!})\ f(\tau^{-1}\left(\frac{k_1}{n}\right).\,\tau^{-1}\left(\left(\frac{n-k_1}{n}\right)\right)e^{-n|\tau(x_1)-\tau(x_2)|}$

$+\sum_{k_1=1}^{n}\tau(x_2))^{k_2}\ \frac{(n)^k}{k!}f(.\,\tau^{-1}(0),\tau^{-1}\left(\frac{k_2-1}{n}\right))e^{-n|\tau(x_2)|}\ +(\tau(x_1))^{n+1}\ f\,(0,1)\ e^{-n|\tau(x_1)|}$

$S_3=\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\tau(x))^k\ \frac{(n)^k}{k!}\ f(\tau^{-1}\left(\left(\frac{k_1}{n}\right)\right).\,\tau^{-1}\left(\left(\frac{k_2}{n}\right)\right)\ e^{-n|\tau(x)|}$

$+\sum_{k_1=1}^{n}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}\ (\frac{(n)^k}{k!})\ f(\tau^{-1}\left(\frac{k_1}{n}\right),0\ )e^{-n|\tau(x_1)-\tau(x_2)|}$

$+\sum_{k_1=0}^{n}\tau(x_2))^{k_2}\ \frac{(n)^k}{k!}f(.\,\tau^{-1}(0),\tau^{-1}\left(\frac{k_2}{n}\right)e^{-n|\tau(x_2)|}+f\,(0,0)\ e^{-n|\tau(x)|}$

Therefore, we have

$S_{n(f;\,\tau(x))}=\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\tau(x))^k\ \frac{(n)^k}{k!}\ f(\tau^{-1}\left(\left(\frac{k_1-1}{n}\right)\right).\,\tau^{-1}\left(\left(\frac{k_2}{n}\right)\right)$

$e^{-n|\tau(x)|}+\sum_{k_1=1}^{n}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}\ (\frac{(n)^k}{k!})\ f(\tau^{-1}\left(\frac{k_1-1}{n}\right).\,\tau^{-1}\left(\left(\frac{n-k_1+1}{n}\right)\right)e^{-n|\tau(x_1)-\tau(x_2)|}$

$+\sum_{k_1=0}^{n}\tau(x_1))^{k_1}\ \frac{(n)^k}{k!}f(\tau^{-1}\left(\frac{k_1-1}{n}\right).\,\tau^{-1}((0)\ )\ e^{-n|\tau(x_1)|}\ +(\tau(x_1))^{n+1}\ f\,(1,0)\ e^{-n|\tau(x_1)|}$

$+\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\tau(x))^k\ \frac{(n)^k}{k!}\ f(\tau^{-1}\left(\left(\frac{k_1}{n}\right)\right).\,\tau^{-1}\left(\left(\frac{k_2-1}{n}\right)\right)$

$e^{-n|\tau(x)|}+\sum_{k_1=1}^{n}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}\ (\frac{(n)^k}{k!})\ f(\tau^{-1}\left(\frac{k_1}{n}\right).\,\tau^{-1}\left(\left(\frac{n-k_1}{n}\right)\right)e^{-n|\tau(x_1)-\tau(x_2)|}$

$+\sum_{k_1=1}^{n}\tau(x_2))^{k_2}\ \frac{(n)^k}{k!}f(.\,\tau^{-1}(0),\tau^{-1}\left(\frac{k_2-1}{n}\right))e^{-n|\tau(x_2)|}\ +(\tau(x_1))^{n+1}\ f\,(0,1)\ e^{-n|\tau(x_1)|}$

$$+\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\tau(x))^k\,\frac{(n)^k}{k!}\,f(\tau^{-1}\left(\left(\frac{k_1}{n}\right)\right).\tau^{-1}\left(\left(\frac{k_2}{n}\right)\right)e^{-n|\tau(x)|}$$

$$+\sum_{k_1=1}^{n}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}\left(\frac{(n)^k}{k!}\right)f\left(\tau^{-1}\left(\frac{k_1}{n}\right),0\right)e^{-n|\tau(x_1)-\tau(x_2)|}$$

$$+\sum_{k_1=0}^{n}\tau(x_2))^{k_2}\,\frac{(n)^k}{k!}f(.\tau^{-1}(0),\tau^{-1}\left(\frac{k_2}{n}\right)e^{-n|\tau(x_2)|}+f(0,0)\,e^{-n|\tau(x)|} \qquad \textbf{2.1}$$

Similarly,

$$S_{n+1(f;\,\tau(x))}=\sum_{k_1=0}^{n+1}\sum_{k_2=0}^{n+1-k_1}e^{-(n+1)\tau(x)}(\tau(x))^k\,\frac{(n+1))^k}{k!}\,(f o\tau^{-1})\left(\frac{k}{n+1}\right)$$

$$=\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\tau(x))^k\,\frac{(n+1))^k}{k!}\,f(\tau^{-1}\left(\left(\frac{k_1}{n+1}\right)\right).\tau^{-1}\left(\left(\frac{k_2}{n+1}\right)\right)e^{-n|\tau(x)|}$$

$$+\sum_{k_1=1}^{n}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}\left(\frac{(n+1))^k}{k!}\right)f(\tau^{-1}\left(\frac{k_1}{n+1}\right).\tau^{-1}\left(\left(\frac{n-k_1+1}{n}\right)\right)e^{-n|\tau(x_1)-\tau(x_2)|}+$$

$$+\sum_{k_1=0}^{n}\tau(x_1))^{k_1}\,\frac{(n+1))^k}{k!}f(\tau^{-1}\left(\frac{k_1}{n+1}\right).\tau^{-1}((0)\,)\,e^{-n|\tau(x_1)|}+(\tau(x_1))^{n+1}\,f(1,0)\,e^{-n|\tau(x_1)|}$$

$$+\sum_{k_2=0}^{n}\tau(x_2))^{k_2}\,\frac{(n+1))^k}{k!}f(.\tau^{-1}(0),\tau^{-1}\left(\frac{k_2}{n+1}\right)e^{-n|\tau(x_2)|}+(\tau(x_2))^{n+1}\,f(0,1)+f(0,0)\,e^{-n|\tau(x)|} \qquad \textbf{2.2}$$

Thus, we can write

$$S_n(f;\tau(x))-S_{n+1}(f;\tau(x))=\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\tau(x))^k\;e^{-n|\tau(x)|}\left\{\frac{(n)^k}{k!}(\,f(\tau^{-1}\left(\left(\frac{k_1-1}{n}\right)\right).\tau^{-1}\left(\left(\frac{k_2}{n}\right)\right))+\right.$$

$$+\frac{(n)^k}{k!}(f(\tau^{-1}\left(\left(\frac{k_1}{n}\right)\right).\tau^{-1}\left(\left(\frac{k_2-1}{n}\right)\right))+\frac{(n)^k}{k!}(f(\tau^{-1}\left(\left(\frac{k_1}{n}\right)\right).\tau^{-1}\left(\left(\frac{k_2}{n}\right)\right))$$

$$-\frac{(n+1))^k}{k!}f(\tau^{-1}\left(\left(\frac{k_1}{n+1}\right)\right).\tau^{-1}\left(\left(\frac{k_2}{n+1}\right)\right)+\sum_{k_1=1}^{n}\left\{\left(\frac{(n)^k}{k!}\right)f(\tau^{-1}\left(\frac{k_1-1}{n}\right).\tau^{-1}\left(\left(\frac{n-k_1+1}{n}\right)\right)\right.$$

$$\left.+\frac{(n)^k}{k!}f(\tau^{-1}\left(\left(\frac{k_1}{n}\right)\right).\tau^{-1}\left(\left(\frac{n-k_1}{n}\right)\right)-\;-\left(\frac{(n+1))^k}{k!}\right)f\left(\tau^{-1}\left(\frac{k_1}{n+1}\right).\tau^{-1}\left(\left(\frac{n-k_1+1}{n}\right)\right)\right)\right\}$$

$$\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}e^{-n|\tau(x_1)-\tau(x_2)|}+\sum_{k_1=0}^{n}\tau(x_1))^{k_1}\left\{\frac{(n)^k}{k!}f(\tau^{-1}\left(\frac{k_1-1}{n}\right).\tau^{-1}((0)\,)e^{-n|\tau(x_1)|}\right.$$

$$+\left.\frac{(n)^k}{k!}\right)f\left(\tau^{-1}\left(\frac{k_1}{n}\right),0\right)\;e^{-n|\tau(x_1)-\tau(x_2)|}\,\frac{(n+1))^k}{k!}+\sum_{k_2=1}^{n}\tau(x_2))^{k_2}\left\{\frac{(n)^k}{k!}f\left(.\tau^{-1}(0),\tau^{-1}\left(\frac{k_2-1}{n}\right)\right)\right.$$

$$\left.+\frac{(n)^k}{k!}f\left(.\tau^{-1}(0),\tau^{-1}\left(\frac{k_2}{n}\right)-\frac{(n+1))^k}{k!}f\left(.\tau^{-1}(0),\tau^{-1}\left(\frac{k_2}{n+1}\right)\right\}e^{-n|\tau(x_2)|}\right\}$$

$$+\{f(1,0)-f(1,0)\}e^{-n|\tau(x_1)|}\,(\tau(x_1))^{n+1}+\{f(0,1)-f(0,1)\}\,(\tau(x_1))^{n+1}\,\{f(0,0)\,f(0,0)\}\,e^{-n|\tau(x)|}$$

$$S_n(f;\,\tau(x))\;-S_{n+1}(f;\,\tau(x))\;=\sum_{k_1=0}^{n-2}\sum_{k_2=0}^{n-2-k_1}\tau(x_1))\tau(x_2))\tau(x))^k\;e^{-n|\tau(x)|}\left\{\frac{(n)^k}{k!}\,(f o\tau^{-1})\left(\frac{k_1}{n},\frac{k_2+1}{n}\right)\right.$$

$$\left.+\frac{(n)^k}{k!}(f o\tau^{-1})\left(\frac{k_1+1}{n},\frac{k_2}{n}\right)+\frac{(n)^k}{k!}(f o\tau^{-1})\left(\frac{k_1+1}{n},\frac{k_2+1}{n}\right)-\frac{(n+1)^k}{k!}(f o\tau^{-1})\left(\frac{k_1+1}{n+1},\frac{k_2+1}{n+1}\right)\right\}$$

$$+\sum_{k_1=0}^{n-1}\tau(x_1))\tau(x_2))\{(f o\tau^{-1})\left(\frac{k_1}{n},\frac{n-k_1}{n}\right)+(f o\tau^{-1})\left(\frac{k_1+1}{n},\frac{n-k_1-1}{n}\right)-(f o\tau^{-1})\left(\frac{k_1+1}{n+1},\frac{n-k_1}{n+1}\right)\}\,\tau(x_1))^{k_1}\tau(x_1))^{n-k_1-1}$$

$$+\sum_{k_1=0}^{n-1}\tau(x_1))\{(f o\tau^{-1})\left(\frac{k_1}{n},0\right)+(f o\tau^{-1})\left(\frac{k_1+1}{n},0\right)-(f o\tau^{-1})\left(\frac{k_1+1}{n+1},0\right)\}\tau(x_1))^{k_1}+\sum_{k_1=0}^{n-1}\tau(x_2))\{(f o\tau^{-1})\left(0,\frac{k_2}{n}\right)+$$

$$(f o\tau^{-1})\left(0,\frac{k_2+1}{n}\right)-(f o\tau^{-1})\left(0,\frac{k_2+1}{n+1}\right)\}\tau(x_2))^{k_2} \qquad \textbf{2.3}$$

Now set,

$I_1 := \frac{(n)^k}{k!}\ (fo\tau^{-1})\left(\frac{k_1}{n}, \frac{k_2+1}{n}\right) + \frac{(n)^k}{k!}(fo\tau^{-1})\left(\frac{k_1+1}{n}, \frac{k_2}{n}\right) + \frac{(n)^k}{k!}(fo\tau^{-1})\left(\frac{k_1+1}{n}, \frac{k_2+1}{n}\right) - \frac{(n+1)^k}{k!}(fo\tau^{-1})\left(\frac{k_1+1}{n+1}, \frac{k_2+1}{n+1}\right)$

$I_2 := (fo\tau^{-1})\left(\frac{k_1}{n}, \frac{n-k_1}{n}\right) + (fo\tau^{-1})\left(\frac{k_1+1}{n}, \frac{n-k_1-1}{n}\right) - (fo\tau^{-1})\left(\frac{k_1+1}{n+1}, \frac{n-k_1}{n+1}\right)$

$I_3 := \{(fo\tau^{-1})\left(\frac{k_1}{n}, 0\right) + (fo\tau^{-1})\left(\frac{k_1+1}{n}, 0\right) - (fo\tau^{-1})\left(\frac{k_1+1}{n+1}, 0\right)$

$I_3 := (fo\tau^{-1})\left(0, \frac{k_2}{n}\right) + (fo\tau^{-1})\left(0, \frac{k_2+1}{n}\right) - (fo\tau^{-1})\left(0, \frac{k_2+1}{n+1}\right)$

We firstly consider I_1, let

$\propto_1 = \frac{k_1+1}{n+1} \geq 0$, $\propto_2 = \frac{k_2+1}{n+1} \geq 0$, $\propto_3 = \frac{n-|k|-1}{n+1} \geq 0$

And

$x_1 = \left(\frac{k_1}{n}, \frac{k_2+1}{n}\right), x_2 = \left(\frac{k_1+1}{n}, \frac{k_2}{n}\right),\ x_3 = \left(\frac{k_1+1}{n}, \frac{k_2+1}{n}\right)$

Then it is easily seen that, $\propto_1 + \propto_2 + \propto_3 = 1$ and $\propto_1 x_1 + \propto_2 x_2 + \propto_3 x_3 = \left(\frac{k_1+1}{n+1}, \frac{k_2+1}{n+1}\right)$. Thus, from the definition of τ – convexity, it readily follows that $I_1 \geq 0$

For I_2 , we set

$\propto_1 = \frac{k_1+1}{n+1} \geq 0$, $\propto_2 = \frac{n-k_1}{n+1} \geq 0$

And

$x_1 = \left(\frac{k_1}{n}, \frac{n-k_1}{n}\right), x_2 = \left(\frac{k_1+1}{n}, \frac{n-k_1-1}{n}\right)$

Thus we have, $\propto_1 + \propto_2 = 1$ and $\propto_1 x_1 + \propto_2 x_2 = \left(\frac{k_1+1}{n+1}, \frac{n-k_1}{n+1}\right)$. Thus, from the definition of τ – convexity, it readily follows that $I_2 \geq 0$

For I_3 , we set

$\propto_1 = \frac{k_1+1}{n+1} \geq 0$, $\propto_2 = \frac{n-k_1}{n+1} \geq 0$

And

$x_1 = \left(\frac{k_1}{n}, 0\right), x_2 = \left(\frac{k_1+1}{n}, 0\right)$

Thus, we have $\propto_1 + \propto_2 = 1$ and $\propto_1 x_1 + \propto_2 x_2 = \left(\frac{k_1+1}{n+1}, 0\right)$. Thus, from the definition of τ – convexity, it readily follows that $I_3 \geq 0$ similarly $I_4 \geq 0$. Therefore, from (2.3)

$s_n(f; \tau(x)) - s_{n+1}(f; \tau(x)) \geq 0$

CONCLUSIONS

We had reached the desired result $s_n(f; \tau(x)) \geq s_{n+1}(f; \tau(x))$, for all $n \in N$

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